# Supplemental Notes

EE503 Week 13 — Dr. Franzke

HW : (none)

① Read : Gubner, Ch. 12

Leon Garcia, Ch. 11

② Daily Derivations (Final Exam prep)

Topics : Markov Chains and Estimation.

① M/M/1 Queue "Birth Death" Process

"Global Balance Equations"

$$\begin{cases} \lambda p_0 = \mu p_1 \\ (\lambda+\mu) p_n = \underbrace{\lambda p_{n-1}}_{\text{"birth"}} + \underbrace{\mu p_{n+1}}_{\text{"death"}} & \text{if } n \geq 1 \end{cases}$$

$\therefore p_n = (1-\rho)\rho^n$ Geometric pdf

② (DTFS) Markov Chain : $\pi_e P = \pi_e$ ← equlibrium pdf.

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \Longrightarrow \pi_e = \left( \frac{b}{a+b}, \frac{a}{a+b} \right)$$

↑ the "google eigenvector"

# M/M/1 Queue

- Queue with 1 server & infinite buffer

  ∴ State space = $\{0, 1, 2, 3, \dots\}$

- First "M": Poisson arrivals (customers)

  with rate $\lambda > 0$ (Poisson process)

  - i.i.d. exponential inter-arrival times

    (renewal process)

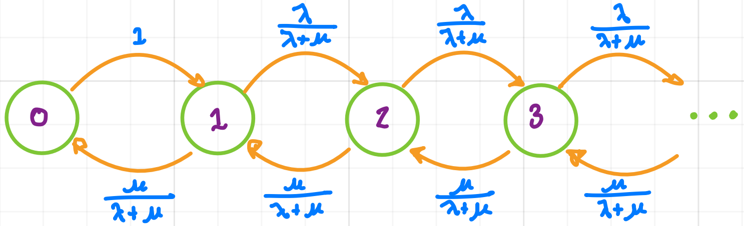
- Second "M": Exponential Service times

  with rate $\mu > 0$

  ∴ pdf $f(t) = \mu e^{-\mu t}$

∴ Continuous Markov Chain: with "jumps"
(Birth-Death process)

"jumps" ⟶ EMBEDDED CHAIN.




∴ $\lambda$: average "birth" rate
$\mu$: average "death" rate

$P_n(t) = \text{Prob}(\text{System is in state } n \text{ at time } t)$

$\therefore$ at equilibrium : $p_n$ = proportion of time system is in state $n$

$$p_n = P_n(\infty) = \lim_{t\to\infty} p_n(t) \quad \text{(if exists)}$$

Assume: $\lambda < \mu$ (else unstable)

Define: $\rho \triangleq \dfrac{\lambda}{\mu}$ $\quad \therefore \; 0 < \rho < 1.$

## Global Balance Equations (memorize)

"birth" "death" At equilibrium

$$\begin{cases} \lambda p_0 = \mu p_1 \\ (\lambda+\mu)\, p_n = \lambda p_{n-1} + \mu p_{n+1} & \text{if } n \geq 1 \end{cases}$$

Will show: $p_n = \left(\dfrac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0$ $\quad$ (pdf: $\sum_{n=0}^{\infty} p_n = 1$)

$\therefore \sum_{n=0}^{\infty} p_0 \cdot \rho^n = \dfrac{p_0}{1-\rho}$

$\therefore p_0 = 1-\rho.$

$$\therefore \boxed{p_n = (1-\rho)\rho^n}$$

Geometric pdf

$\therefore P(N \geq n) = \rho^n$

$\therefore E[N] = \boxed{\dfrac{\rho}{1-\rho}} = \dfrac{\lambda}{\mu-\lambda}$ $\quad$ expected number of customers in the system

$V[N] = \boxed{\dfrac{\rho}{(1-\rho)^2}}$

## Derivation

**Assume:**

arrival

- $P(1 \overset{B}{\text{BIRTH}} \text{ in } (t, t+\Delta t)) \approx \lambda \cdot \Delta t + o(\Delta t)$
- $P(\geq 2 \text{ BIRTHS in } (t+\Delta t)) \approx o(\Delta t)$
- $P(1 \overset{D}{\text{DEATH}} \text{ in } (t+\Delta t)) \approx \mu \cdot \Delta t + o(\Delta t)$

service

for $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$

$\therefore\ c\,o(\Delta t) = o(\Delta t)$

$o(\Delta t) + o(\Delta t) = o(\Delta t)$

Put $p_n(t) = P(\overset{N(t)}{\text{population}} = n \text{ at time } t)$ for $n \geq 0$

$\left( \begin{array}{l} \lambda_n = \lambda \quad \forall n, \text{ but} \\ \mu_0 = 0 \ \& \ \mu_n = \mu \text{ for } n \geq 1 \end{array} \right)$

### Case 1: $n \geq 1$

4 birth-death (D-B) cases plus $o(\Delta t)$

B: 0 or 1
and
D: 0 or 1

$$\therefore p_n(t+\Delta t) = \underbrace{p_n(t)(1-\lambda\Delta t)(1-\mu\Delta t)}_{B=0\ \&\ D=0} + \underbrace{p_n(t)\cdot(\lambda\Delta t)(\mu\Delta t)}_{B=1\ \&\ D=1}$$

$$+ \underbrace{p_{n-1}(t)(\lambda\Delta t)(1-\mu\Delta t)}_{B=1\ \&\ D=0} + \underbrace{p_{n+1}(t)\cdot(1-\lambda\Delta t)(\mu\Delta t)}_{B=0\ \&\ D=1} + o(\Delta t)$$

$$= p_n(t)\left[1 - \lambda\Delta t - \mu\Delta t + \lambda\mu(\Delta t)^2\right] + p_n(t)\lambda\mu(\Delta t)^2$$

$$+ p_{n-1}(t)(\lambda\Delta t - \lambda\mu(\Delta t)^2) + p_{n+1}(t)(\mu\Delta t - \lambda\mu(\Delta t)^2) + o(\Delta t)$$

$$= p_n(t)(1 - \lambda\Delta t - \mu\Delta t) + p_{n-1}(t)\lambda\Delta t + p_{n+1}(t)\mu\Delta t$$

$$+ \underbrace{\left[2p_n(t)\lambda\mu - p_{n-1}(t)\lambda\mu - p_{n+1}(t)\lambda\mu\right](\Delta t)^2}_{= o(\Delta t) \text{ since } \lim_{\Delta t \to 0} \frac{(\Delta t)^2}{(\Delta t)} = \lim_{\Delta t \to 0} \Delta t = 0} + o(\Delta t)$$

$$= p_n(t)\left(1-\lambda\Delta t-\mu\Delta t\right) + p_{n-1}(t)\lambda\Delta t + p_{n+1}(t)\mu\Delta t + o(\Delta t)$$

since $o(\Delta t)+o(\Delta t)=o(\Delta t)$

$$= p_n(t) - p_n(t)(\lambda+\mu)\Delta t + p_{n-1}(t)\lambda\Delta t + p_{n+1}(t)\mu\Delta t + o(\Delta t)$$

$$\therefore \dot{p}_n(t) = \frac{dp_n(t)}{dt} = \lim_{\Delta t\to 0}\frac{p_n(t+\Delta t)-p_n(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{-p_n(t)(\lambda+\mu)\cancel{\Delta t} + p_{n-1}(t)\lambda\cancel{\Delta t} + p_{n+1}(t)\mu\cancel{\Delta t}}{\cancel{\Delta t}} + \underbrace{\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}}_{=0}$$

$$= -p_n(t)(\lambda+\mu) + p_{n-1}(t)\lambda + p_{n+1}(t)\mu.$$

$\therefore$ At equilibrium: $\dot{p}_n(t)=0$

$$\longleftrightarrow \quad \underline{(\lambda+\mu)p_n = \lambda p_{n-1} + \mu p_{n+1}}$$

## Case 2 : $n=0$

(i.e. $p_{-1}(t)=0$)

3 B-D cases to $(\Delta t)$ since can't have $B=0$ & $D=1$ $\therefore \mu_0=0$

$$\therefore p_0(t+\Delta t) = p_0(t)\underbrace{(1-\lambda\Delta t)}_{B=0}\underbrace{(1-\mu_0\Delta t)}_{D=0} + p_0(t)\underbrace{(\lambda\Delta t)}_{B=1}\underbrace{(\mu\Delta t)}_{D=1}$$

$$+ p_1(t)\underbrace{(1-\lambda\Delta t)}_{B=0}\underbrace{(\mu\Delta t)}_{D=1} + o(\Delta t)$$

$$= p_0(t)\left(1-\lambda\Delta t-\mu_0\Delta t+\lambda\mu_0(\Delta t)^2\right) + p_0(t)\lambda\mu(\Delta t)^2$$

$$+ p_1(t)\mu\Delta t - p_1(t)\lambda\mu(\Delta t)^2 + o(\Delta t)$$

$$= p_0(t) - p_0(t)(\lambda + \mu_0)\Delta t + \underbrace{p_1(t)\mu\Delta t + \lambda\mu\left(p_0(t) - p_1(t)\right)(\Delta t)^2}_{o(\Delta t)} + o(\Delta t)$$

$$= p_0(t) - (\lambda + \underbrace{\mu_0}_{=0})\, p_0(t)\Delta t + \mu_1 p_1(t)\Delta t + o(\Delta t)$$

$$= p_0(t) - \lambda p_0(t)\Delta t + \mu_1 p_1(t)\Delta t + o(\Delta t)$$

$$\therefore\ 0 = \dot{p}_0(t) = \lim_{\Delta t \to 0} \frac{p_0(t + \Delta t) - p_0(t)}{\Delta t}$$

$$= \lim_{n \to \infty} \frac{-p_0(t)\lambda\cancel{\Delta t} + p_1(t)\mu\cancel{\Delta t}}{\cancel{\Delta t}} + \lim_{n\to\infty} \frac{o(\Delta t)}{\Delta t}$$

$$= -\lambda p_0(t) + \mu p_1(t)$$

$$\therefore\ \dot{p}_0(t) = 0 \longleftrightarrow \lambda p_0 = \mu p_1$$

QED - Global Balance Equations

**Thrm:** (for Global Balance Equations) $p_n = \left(\frac{\lambda}{\mu}\right)^n p_0$

Prf: (by induction on $n$)

Basis step: $n=1$ ($\&$ $n=2$)

For $n=1$: $p_0\lambda = p_1\mu \longleftrightarrow p_1 = \left(\frac{\lambda}{\mu}\right)^1 p_0$

For $n=2$: $p_1(\lambda+\mu) = p_0\lambda + p_2\mu$

$$\therefore\ p_2 = \frac{1}{\mu}\left[p_1(\lambda+\mu) - p_0\lambda\right] \overset{n=1}{=} \frac{1}{\mu}\cdot\left[\left(\frac{\lambda}{\mu}\right)p_0(\lambda+\mu) - p_0\lambda\right)$$

$$= \frac{1}{\mu}\left(\frac{\lambda^2}{\mu}p_0 + \cancel{\lambda p_0} - \cancel{\lambda p_0}\right) = \left(\frac{\lambda}{\mu}\right)^2 \cdot p_0$$

QED: Basis

Induction step:

<u>Assume</u> Induction Hypothesis (IH): $p_n = \left(\frac{\lambda}{\mu}\right)^n p_0$

Show: $p_{n+1} = \left(\frac{\lambda}{\mu}\right)^{n+1} p_0$

$p_{n+1}\cdot\mu = p_n(\lambda+\mu) - p_{n-1}\lambda$ from G.B. $(\lambda+\mu)p_n = \lambda p_{n-1} + \mu p_{n+1}$

$$\therefore\ p_{n+1} = \left(\frac{\lambda+\mu}{\mu}\right)p_n - \left(\frac{\lambda}{\mu}\right)p_{n-1}$$

$$\overset{\text{I.H.}}{=} \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n p_0 - \left(\frac{\lambda}{\mu}\right)p_{n-1}$$

$$\overset{\text{I.H.}}{=} \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n p_0 - \left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n-1} p_0$$

$$= \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n p_0 - \left(\frac{\lambda}{\mu}\right)^n p_0$$

$$= \left[\frac{\lambda+\mu}{\mu} - 1\right]\left(\frac{\lambda}{\mu}\right)^n p_0$$

$$= \left[\frac{\lambda}{\mu} + \cancel{1} - \cancel{1}\right]\left(\frac{\lambda}{\mu}\right)p_0 = \left(\frac{\lambda}{\mu}\right)^{n+1} p_0$$

QED

$\therefore\ p_n = \rho^n p_0$ if $\rho = \frac{\lambda}{\mu}$

$$\therefore\ 1 \overset{\text{pdf}}{=} \sum_{n=0}^{\infty} p_n \overset{\text{thm}}{=} \sum_{n=0}^{\infty} \rho^n p_0 = p_0 \cdot \sum_{n=0}^{\infty} \rho^n$$

$$\overset{\text{G.S.}}{=} p_0 \cdot \frac{1}{1-\rho}$$

$$\therefore\ p_0 = 1-\rho$$

geometric structure

$$\therefore\ \boxed{p_n = \rho^n(1-\rho)}$$

$$\therefore\ P(N \geq n) = \sum_{k=n}^{\infty} p_k \overset{\text{Thm}}{=} \sum_{k=n}^{\infty} (1-\rho)\rho^k = (1-\rho)\sum_{k=n}^{\infty} \rho^k$$

$$\overset{\text{G.S.}}{=} (1-\rho)\frac{\rho^n}{1-\rho} = \rho^n$$

$$\therefore\ P(N \geq n) = \rho^n.$$

**Ex:** In-n-Out Burger drivethru $(M/M/1)$

Say 3 customers arrive per minute (late night): $\lambda = 3$

Find service time $\mu$ so that 95% of time queue has fewer than 10 customers

$$\therefore\ P(N \geq 10) = \rho^{10} = \left(\frac{\lambda}{\mu}\right)^{10} = \left(\frac{3}{\mu}\right)^{10}$$

$$\therefore\ 0.95 = P(N < 10) = 1 - P(N \geq 10) = 1 - \left(\frac{3}{\mu}\right)^{10} = \frac{19}{20}$$

$$\therefore\ \left(\frac{3}{\mu}\right)^{10} = \frac{1}{20} \longleftrightarrow \mu = 3\sqrt[10]{20} \approx 4.048 \text{ customers per minute.}$$

**Recall:** $|a| < 1 \longrightarrow \displaystyle\sum_{k=n}^{\infty} a^k \overset{\text{G.S.}}{=} \frac{a^n}{1-a}$

$$\therefore \sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2} \qquad \left( = \sum_{n=1}^{\infty} n a^n \right)$$

$$\therefore \sum_{n=0}^{\infty} n^2 a^n = \frac{a+a^2}{(1-a)^3} \qquad \left( = \sum_{n=1}^{\infty} n^2 a^n \right)$$

$$\therefore E_N[N] = \sum_{n=0}^{\infty} n \cdot p_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n$$

$$\overset{\text{G.S.}}{=} (1-\rho) \cdot \frac{\rho}{(1-\rho)^2} = \boxed{\frac{\rho}{1-\rho}} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}.$$

$$E_N[N^2] = \sum_{n=0}^{\infty} n^2 \cdot p_n = \sum_{n=0}^{\infty} n^2 \cdot \rho^n (1-\rho) = (1-\rho)\sum_{n=0}^{\infty} n^2 \rho^2$$

$$\overset{\text{G.S.}}{=} (1-\rho) \cdot \frac{\rho+\rho^2}{(1-\rho)^3} = \frac{\rho+\rho^2}{(1-\rho)^2}$$

$$\therefore V_N[N] = E_N[N^2] - E_N^2[N] = \frac{\rho+\cancel{\rho^2}}{(1-\rho)^2} - \frac{\cancel{\rho^2}}{(1-\rho)^2} = \boxed{\frac{\rho}{(1-\rho)^2}}$$

$$\therefore E_N[N] = \frac{\rho}{1-\rho} \quad \text{and} \quad V_N[N] = \frac{\rho}{(1-\rho)^2}$$

**Ex:** from above: $\lambda = 3$ and $\mu \approx 4$ $\qquad \therefore \rho = 3/4$

$$\therefore E_N[N] = \frac{\rho}{1-\rho} = \frac{3/4}{1-3/4} = \frac{3/4}{1/4} = 3 \text{ customers}$$

$$V_N[N] = \frac{\rho}{(1-\rho)^2} = \frac{3/4}{(1-3/4)^2} = \frac{3/4}{1/16} = \frac{3 \cdot 16}{4} = 12$$

$$\therefore \sigma_N = \sqrt{V_N[N]} = \sqrt{12} = 2\sqrt{3} \text{ customers.}$$

# Markov chains

Markov idea : Future = f(Present)

does not depend on the Past.

$$f(x_{n+1} \mid x_1, \dots, x_n) = f(x_{n+1} \mid x_n)$$

Multiplication Theorem:

always holds

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \dots \cap A_{n-1})$$

Independence: $P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P(A_k)$

Markovity lies in between

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_2) \cdots P(A_n \mid A_{n-1})$$

DTFS: Discrete Time and Finite State

State: $S_1, S_2, \dots, S_m$

Ex:
- Frog jumping from lily pad to lily pad ("random walk")
- Chess-board configuration
- Google "Page rank" algorithm (Larry page)
  ↳ US Patent # 6,285,999 B1.

page rank ⟷ main eigenvector from Markov-chain matrix.

**Defn:** $X_1, X_2, \ldots$ is a Markov Chain iff

$$P(X_{k+1} = x_{k+1} \mid X_1 = x_1, \ldots, X_k = x_k) = P(X_{k+1} = x_{k+1} \mid X_k = x_k) \quad \forall k$$

**Ex.** iid sum $S_n = \sum_{k=1}^{n} X_k$ is a Markov Chain

(infinite state space)

**Prf:** (SIT technique)

$$P(S_{n+1} = s_{n+1} \mid S_1 = s_1, \ldots, S_n = s_n)$$

$$= P(X_{n+1} + S_n = s_{n+1} \mid S_1 = s_1, \ldots, S_n = s_n)$$

$$\overset{\text{Sub}}{=} P(X_{n+1} + s_n = s_{n+1} \mid S_1 = s_1, \ldots, S_n = s_n)$$

$$= P(X_{n+1} = s_{n+1} - s_n \mid S_1 = s_1, \ldots, S_n = s_n)$$

$$\overset{\text{ind}}{=} P(X_{n+1} = s_{n+1} - s_n)$$

by defn of $S_n$

$$= P(X_{n+1} + s_n = s_{n+1})$$

$$\overset{\text{Sub}}{=} P(X_{n+1} + S_n = s_{n+1} \mid S_n = s_n)$$

$$= P(S_{n+1} = s_{n+1} \mid S_n = s_n)$$

$\therefore \{S_n\}$ is a Markov Chain

QED.

**History:** Andrei Markov - 1906 (St. Petersburg)

- Markov wanted to show that independence was not needed for the LLN and CLT

  He succeeded: there are Markov LLN and CLT.

- 1913: Analyzed 20,000 letters of Pushkins poem

  Eugeny Onegin

  found: $p(\text{vowel}) = 0.432$ ← stationary value

  $p(\text{vowel} \mid \text{vowel just occured}) = 0.128 = P(v \to v)$

  $p(\text{vowel} \mid \text{consonant just occured}) = 0.663 = P(c \to v)$

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**Simplest case:** DTFS

n-states : ①, ②, ..., ⓝ (lily pads or web pages)

$p_{ij} = P(i \to j) = P(X_{k+1} = j \mid X_k = i)$

↑ "transition probability"

**Time-Homogeneous:** $P(X_{k+1} = j \mid X_k = i) = p_{ij}$ for $\forall k$.

↑ $p_{ij} \geq 0$

# Transition Matrix P:

$$\underset{n\times n}{P} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix}$$

← each row is a pdf
∴ stochastic matrix.

infinite matrix for infinite state Markov chain

So that each row sums to 1:

$$\sum_{k=1}^{n} P_{jk} = 1 \quad \text{for } \forall j.$$

∴ $P: S^n \longrightarrow S^n$ maps probability vector $\pi$ to probability vector

n-simplex

∴ P has a fixed point pdf $\pi$:

$$\boxed{\pi P = \pi}$$

(from Brouwer's fixed point theorem since P is linear and continuous and since $S^n$ is convex and compact)
closed and bounded

∴ $\underset{1\times n}{\pi}$ is an eigenvector with eigenvalue $\lambda = 1$

Fact: All other eigenvalues $\lambda'$: $|\lambda'| < 1$.
↳ from Perron-Frobenius Theorem for non-negative matrices.

For any pdf $u$: $\quad u(k) = u(0)P^k$
for time-evolution $u(0), u(1), \ldots, u(k)$
↳ from Chapman-Kolmogorov equations)

**Defn:** $\underset{n\times n}{P}$ is <u>irreducible</u> (regular) if $\exists k>0$: $P^k$ is a positive matrix: $p_{ij}^{(k)} > 0 \quad \forall i \, \forall j$

cannot happen if $P$ has a 1 on a main diagonal unless $\underset{1\times 1}{P} = 1$. $\therefore p_{kk} = 1 \longrightarrow \sim$irreducible

**Defn:** $P$ is <u>aperiodic</u> iff the number of steps between any two states (i) and (j) is an integer multiple (not forced into fixed cycle).

Note: $P$ is aperiodic if no eigenvalue $\lambda'$: $\lambda' = -1$

**Thrm:** ("AI") If the transition matrix $\underset{n\times n}{P}$ is (1) aperiodic and (2) irreducible then $P$ has a stationary pdf $\pi_e$:

unique since finite n

(1) $\pi_e P = \pi_e$

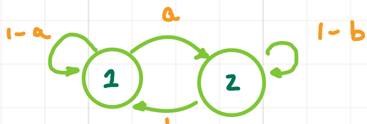
(2) $\lim_{k\to\infty} P^k = \begin{pmatrix} -\pi_e- \\ -\pi_e- \\ \vdots \\ -\pi_e- \end{pmatrix}$ $\therefore$ in the long run

(3) $\lim_{k\to\infty} uP^k = \pi_e$ for $\forall u \in S^n$.

<u>Basis of MCMC approximation</u> (Gibbs sampler)

→ identify problem with solution $= \pi_e$.

**Ex:** Special case: $2\times2$ $P$




$$\underset{2\times2}{P} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array}$$ for $a, b \in [0,1]$.

☆ $\therefore$ Proposition: $\pi_e = \left( \frac{b}{a+b}, \frac{a}{a+b} \right)$ if $P$ is $2\times2$

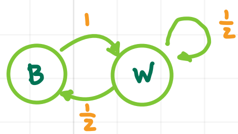
**Prf:**
$$\pi_e P = \left( \frac{b}{a+b}, \frac{a}{a+b} \right) \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$
$$= \left( \frac{b - \cancel{ab} + \cancel{ab}}{a+b}, \frac{\cancel{ab} + a - \cancel{ab}}{a+b} \right)$$
$$= \left( \frac{b}{a+b}, \frac{a}{a+b} \right) = \pi_e \quad \therefore \text{fixed point.}$$

QED

**Ex:** Suppose each day John walks (W) or bikes (B) to class. If John bikes one day then he never bikes the next day. If John walks one day then the next day he is equally likely to bike or walk.

Q: In the long run... how often does John bike to class?

A: $$P = \begin{array}{c c} & \begin{array}{cc} B & W \end{array} \\ \begin{array}{c} B \\ W \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$ from above facts




$$= \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

Note: $\lambda_1 = 1$ and $\lambda_2 = 1 - a - b = -\frac{1}{2} \neq 1$ $\quad \therefore P$ is aperiodic (A)

$$P^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$ $\therefore$ all positive entries

$\therefore$ $P$ is irreducible (I)

$\therefore$ Theorem applies for unique equilibrium pdf: $\pi_e$.

$$\pi_e = \left( \overset{B}{\frac{b}{a+b}}, \overset{W}{\frac{a}{a+b}} \right) = \left( \frac{1/2}{3/2}, \frac{1}{3/2} \right) = \left( \overset{B}{1/3}, \overset{W}{2/3} \right)$$

$\therefore$ Bike $1/3$ the time.

Check: $$\pi_e P = \left(\frac{1}{3}, \frac{2}{3}\right) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left( \frac{2}{3}\cdot\frac{1}{2}, \frac{1}{3} + \frac{2}{3}\cdot\frac{1}{2} \right)$$
$$= \left(\frac{1}{3}, \frac{2}{3}\right)$$
$$= \pi_e$$

Extension: Hidden Markov Model (HMM)
- train with E-M algorithm.

Ex: Google Page Rank Algorithm "random surfer on N web pages"

$r(A)$ = rank of page A

$$= \frac{\alpha}{N} + (1-\alpha)\cdot\left( \frac{r(B_1)}{|B_1|} + \cdots + \frac{r(B_n)}{|B_n|} \right)$$ $\therefore \sum r(A) = 1$

where $B_1, \ldots, B_n$ are the "backlink" pages of A with ranks $r(B_1), \ldots, r(B_n)$

- $\alpha \in [0, 1]$
- $|B_k|$ = # "forward links" of $B_k$
- $N$ = total # of pages on the web

$\alpha \approx P(\text{surfer jumps randomly to any web page})$

$\alpha$ = "damping factor" ← keeps algorithm from getting stuck in loops.

$\approx 0.15$

matrix of all 1's

Put $\underset{N\times N}{P} = \frac{\alpha}{N} \underset{N\times N}{\mathbb{1}} + (1-\alpha) \underset{N\times N}{B}$ where

- $b_{ij} = \begin{cases} \frac{1}{n_i} & \text{if node } i \text{ points to node } j \\ 0 & \text{else} \end{cases}$

- $n_i$ = total # of forward links from $i$

$\therefore \exists$ unique $\underset{1\times N}{\pi_e}$ : $\pi_e P = \pi_e$ since $P$ is aperiodic and irreducible.

$\therefore \pi_e = p_\infty$ is Google ranking of all $N$ webpages

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Markov chains $\{X_k\}$ obey forms of LLNs and CLTs if they are suitaby "ergodic" (aperiodic and irreducible and positive recurrent) - even though the chain $X_1, X_2, \ldots$ is NOT independent.

## Markov Law of Large Numbers (MCLLN)

If
- $\pi$ is the stationary pdf
- Markov chain $X_1, X_2, \ldots$ is (Harris) ergodic
- $f : X \longrightarrow \mathbb{R}$ is measureable
- $E_\pi[\,|f|\,] < \infty$

Then with probability one (˘o˘):

$$\bar{f}_n(x) = \frac{1}{n} \cdot \sum_{k=1}^{n} f(x_k) \xrightarrow{\circ} E_\pi[f(x)]$$

With these and further technical conditions:

MC CLT:

$$\sqrt{n}\,(\bar{f}_n(x) - E_\pi[f]) \xrightarrow{d} W \sim N(0, \sigma_f^2)$$

if $\sigma_f^2 < \infty$ but now $\sigma_f^2$ has the form:

$$\sigma_f^2 = V_\pi[f(x_1)] + \mathrm{Cov}_\pi[f(x_1), f(x_k)].$$

∴ Markov was right: independence is not necessary for LLN or CLT.